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CONTENTS

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THE MATHEMATICS TEACHER

Volume XXXII Number 8

Edited by William David Reeve

Mathematics and the Social Sciences*

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THERE are many reasons why teachers of the social sciences should be interested in mathematics, quite aside from such technical parts of the subject as their statistical needs might lead them to. In the present note I want to give some general indications of the way in which mathematical concepts, rather than mathematical technique, bear directly upon points of view which seem to be of fundamental importance for the objectives pursued by the teaching of the social sciences in the schools and in the undergraduate colleges. I take it that the contribution which these subjects wish to make to the education of our young people consists principally in the insight they give into the relations between individuals and groups of human beings. For example, the study of history reveals the gradual development of states, their struggles for control over the lives of individuals and of smaller groups which sought to maintain special privileges. Sociology studies the relations between man and his social environment, the formation of associations to facilitate the satisfaction of human wants. Political science is concerned with the forms in which agencies for carrying on the activities of groups of men have been cast. It is with such fundamental aspects of the social sciences that mathematical concepts have important connections. Since all except the future specialist are primarily concerned with these fundamental aspects, rather than with a study of details, the connecting of mathematics and the social sciences is of importance to the schools. In support of the general thesis which has here been outlined, three topics will be discussed, viz., *order, inverse functions* and *existence theorems.*

I. ORDER

There are probably not many high school teachers of mathematics who have not met students who insisted upon placing an equality sign between $(a+b)^2$ and a^2+b^2, or between $\sqrt{a^2+b^2}$ and $a+b$, or between $(a^2+b^2)/(a+b)$ and $a+b$. "Why not?", such a student may, and should ask, "you say that $\sqrt{a^2b^2}=ab$, and that $(a^2b^2)/ab=ab$. What difference is there between these things you allow me to do and the others which are forbidden?" A good question, one that gives the teacher an excellent opportunity to prove his worth. It can be answered in a variety of ways. The suitability of the answer depends somewhat on the purpose one wishes to accomplish.

For our present purpose, it is best to

* A paper read at the San Francisco Meeting of the National Council of Teachers of Mathematics, and the Department of Secondary Education of the N.E.A., July 5, 1939.

lead the student to take a closer look at these things. Indeed (and here I shall limit the discussion to only one instance; the reader will readily see how to deal with others), it is true that $(a+b)^2$ and a^2+b^2 are very similar, not only in appearance, but in content as well. In both these expressions, two numbers are involved and two operations, viz. adding and squaring; where is the difference between them? In the first, the addition takes place first, the squaring of the result afterwards; in the second expression, we are asked first to square each of the two numbers, then to add the results. Thus the essential difference between the two expressions lies in the *order* in which the two operations, adding and squaring, are carried out.

Next the student can be led to see that between $(ab)^2$ and a^2b^2 the difference is of the same character; it lies in the order in which two operations are to be carried out. But this time the operations are multiplying and squaring. The important fact which the study of these various operations reveals is that, whereas a change in the order in which multiplication and squaring are performed does not affect the result, this is not true of adding and squaring; for these two operations a change of order does affect the result. Is this at all surprising, or is it similar to common experience? Such homely illustrations as that of putting on stockings and shoes, or coat and hat, should be effective in driving home the observation that a change in the order in which acts are performed does affect their result sometimes, but not always. We learn to distinguish between such situations quite easily, without special instruction. We hardly ever meet a person, whose shoes are inside his stockings, even though he may have put his hat on before or after he donned his coat. The wisdom which guides us in such matters is not restricted to those who have had the "advantages of an education."

Not all situations in which we have to decide whether or not the order in which to do certain things has to be taken into account, are quite as simple as those that have been mentioned; the mathematical examples are very simple. Throughout our common experience we meet problems of this character: Does it make any difference whether we buy the automobile (or the radio, or the refrigerator, or the house) first and then try to earn the money to pay for it, or whether we earn the money before buying? Does it make any difference whether we think first and then act, or should we act first? Should a dispute between nations be submitted to a test of armed strength, to be followed by a "peace conference," or should the conference take place first? Such questions are perhaps not capable of a general answer. It is however of first rate importance that they should be considered. The value of the simple mathematical situation with which we started this discussion of order lies in the fact that it gives the teacher of mathematics, even on the elementary level, an unequalled opportunity to bring this importance into the foreground. That similar opportunities present themselves in later stages should be evident to any one who has thought about the general concept of linear operators, or about the facts which are expressed in the formulae $\sin(A+B)\neq\sin A+\sin B$, $D_x(u+v)=D_xu+D_xv$, $D_x(uv)\neq D_xu.D_xv$. In the very abstractness of these relations lies one of their values, because they can thus serve to emphasize the importance of order, unaffected by prejudices of any sort, whether philosophical, religious or political. Thus, rather than berating the pupil who does not hesitate to replace $\sqrt{a^2+b^2}$ by $a+b$, the teacher should recognize the opportunity this student offers him to link his subject with human experience and to introduce a concept of general social significance. The social science teacher would then not fail to recognize the value of the contributions made by his colleagues in mathematics.

II. Inverse Functions

When a child has learned that $5+7=12$, he has established a relation between the numbers 5, 7 and 12; in this relation 12 occupies a special position. We may say that it is the subject of the sentence: "Twelve is the result of adding seven to to five," which is represented above in symbolic form, in the decimal notation. Just as it is a useful grammatical exercise to change a sentence from the active to the passive voice, so it is also a valuable exercise to restate our sentence in such a way that "seven" or "five" become the subject, thus, "Seven is the result of subtracting five from twelve," $7=12-5$. We gain insight into the relation between John and his dog when we change the sentence "John beats the dog" to the form "The dog is beaten by John." In similar manner we gain insight into the relations between 5, 7 and 12, when we change $5+7=12$ to the forms $5=12-7$ and $7=12-5$.

It is a commonplace among mathematicians that relations between two or more variables have not been completely understood until they have been looked at from the point of view of each of the variables involved. From $y=\sin x$, we pass to $x=\text{arc}\sin y$; from $r=e^t$ to $t=\log_e r$, from $u=D_x y$ to $y=\int u\,dx$, from elliptic integrals to elliptic functions, etc. As must be clear from the first illustration however, we do not have to wait until the college subjects are reached to illustrate the concept of the inverse function. For the teacher who has learned to use his eyes and who has exercised this faculty, the road he has to travel, presents at every stage, from the lowest level to the highest, material which contributes to the building up of this concept.

Of what value is this concept in human relations? John and his dog suggest the answer. Is it fantastic to say that conflicts between two or more parties can be solved satisfactorily only after the points at issue have been studied from the point of view of each of the interested groups? Is it important to learn that social questions which affect two or more individuals, or two or more groups, must be considered from the standpoint of each of them? Is it unreasonable to say that we can not hope to attain a *solution* of the capital-labor problem, nor of the C.I.O.-A.F. of L. controversy, nor of the rural-urban conflict of interests, nor of the numerous industry-agriculture difficulties, until each side in these various contests has learned to look at the scene of operations from the point of view of his opponent as well as from his own? The persistent introduction of inverse functions as an essential step in the understanding of functional relations, which is characteristic of much of modern mathematics, gives the teacher of this subject opportunities of incomparable value to the social sciences.

III. Existence Theorems

It is unfortunate that so little attention is paid in the early stages of mathematical education to the question as to the existence of a solution of the problems with which the pupil is asked to deal, and still less to the question of the uniqueness of the solution. It is perhaps a result of the excessive use of textbooks that pupils rarely meet a problem which has no solution. Even in dealing with "radical equations," which furnish such a good opportunity for their introduction, equations which do not possess a solution are usually either avoided or else put in a form in which the ambiguous meaning of the radical obscures the real situation. There is no doubt that the rational operations would gain a good deal in the interest of young children, if they were told (but in terms which they would appreciate!) that these operations are uniquely possible, almost without exception. The way would then be prepared for the very important exception.

The children who had been introduced to this fundamental property of the system of rational numbers, would then approach the linear equation $ax+b=0$ in a different spirit. Before learning the tech-

nique by means of which a solution is found, they would want to know whether a solution exists. They would appreciate the statement that if $a \neq 0$ and only in that case, the equation $ax+b=0$, in which a and b are integers, will possess exactly one solution in the set of rational numbers. They would acquire the habit of asking, before using a technical procedure which is supposed to lead to the solution of a problem, under what conditions and in what sense a solution exists. It is wrong to say that x^2+1 is not factorable; if we admit linear factors with complex coefficients, it is factorable, but not if only linear factors with real coefficients are admitted. On the other hand, there exist no linear factors of the function x^2+y^2-1, even if complex coefficients are admitted.

Associated with every technical process which is learned in mathematics, there should be an existence theorem, which states the exact conditions under which the process is effectively applicable. This is quite generally recognized in advanced parts of the subject. There is no reason, in my judgment, why it should not be adopted for the earlier stages. This does not mean of course that the elementary work should include proofs as well as statements of existence theorems. It is important that the students be led to recognize at as early a stage as possible the importance of such theorems. Perhaps it will be thought that in making this demand I overstep the bounds of reasonableness; in such case, I would not insist. But I would insist that every teacher of mathematics should know for every type of problem that he asks his pupils to solve in great quantities, the exact conditions under which a solution exists and is unique.

Curiously enough, the social sciences seem to be completely unaware of the desirability of existence theorems. Many problems are attacked without consideration of the question whether a solution exists, too often without a clear statement of when the problem may be considered as having been solved. Schemes are proposed to relieve unemployment, plans are brought forward for providing social security, and methods for securing world peace. We are urged to use this method and that to bring about a more reasonable way of life. It is quite certain that in all such proposals, approximate solutions would be gratefully accepted, but it would be useful to know the type of solution that is hoped for.

Perhaps the empirical method is the only one by which the social sciences can advance. It is a costly procedure; moreover social experimentation is in essential ways different from experimentation in the natural sciences. Under these circumstances the concept of an existence theorem should be of great value. Much would be gained, for instance, if it were understood that for the attainment of world peace, certain conditions as to social and economic organization are indispensable, if it were known whether or not the unemployment problem is capable of a complete solution. If we had considered the question whether free trade is a necessary condition for the attainment of world peace, whether collective ownership of the means of production is a necessary condition for social security, there might be fewer proposals for "social betterment," but also less disillusionment, fewer blasted hopes.

The consideration of the existence and the uniqueness of solutions should be of great value to the social scientist in the study of his problems. Upon the teacher of mathematics, rests the obligation of treating his subject in such a manner as to make it yield to the utmost the rich harvest of broad human values of which it contains the abstract basis. Upon school and college administrators rests the obligation of providing the best possible conditions for the liberation and complete functioning of these values.

From Table to Graph to Formula

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In many of the modern texts on Second Course Algebra there is included, just after the section on simultaneous, first degree equations in two unknowns, a topic dealing with the derivation of a formula from a table of related values. A typical example is given here:

If when $C=6\ \ 8\ \ 10\ \ 12$
$D=2\ \ 5\ \ \ 8\ \ 11$

find the equation, if any, connecting C and D. The answer is $3C-2D=14$, which is stated at this point merely to show the *form* of the final equation required.

The older texts do not contain this section. It has arisen lately as a live topic in an algebra which stresses the formula, the graph, the equation—the function concept. High school pupils belong to a new day now. Integrated programs, subject-matter, adjustments, dead-wood eliminations, modern-usage conformities—all have contributed to new emphasis and better teaching methods at strategic points in the mathematical course. This may explain why it is true that the more recent the text, the more space is given to the above topic. Moreover, the growing interest in "fusion mathematics" may have something to do with the introduction, retention and interest-holding status of the section under discussion.

Some teachers have regarded the table-of-related-values topic as detached and of minor importance. How can this topic, they contend, be an integral part of the course? What use can be made of it? Of course, these questions may be the result of a wrong viewpoint, or perhaps there is not sufficient enthusiasm to carry pupils far enough to see the real significance of the subject. The necessary consequence of such an attitude has been an unfortunate tendency on the pupil's part to use a mechanical rule-of-thumb method here and to give the topic only a very superficial study. Alas! He must "drink deep or taste not the Pierian Spring."

The recognized method of solving the problem submitted in the first paragraph is as follows: (1) Plot the points (6, 2), (8, 5,) etc., on squared paper, using suitable units and axes. (2) If they are on a straight line, then the variables C, D are connected by the general formula $C=mD+b$ or $D=mC+b$ where m and b are constants. More of this later. (3) Set up two equations in m and b by substitution from the table thus: $6=2m+b$ and $8=5m+b$. (4) Solve these equations for m and b. (5) Substitute these values in the general formula in (2). Remove fractions and bring C and D to the same side of the resulting equation. (6) Check for correctness by reference to each pair of corresponding values in the table. The pupil has thus gone from table to graph to formula.

Some of the difficulties inherent in the solution will now be considered. Perhaps they are more apparent than real. (1) Teachers tell us that "the plotting of the points takes time and care and that therefore the pupils do the first step half-heartedly." The unambitious, like the poor, are always with us. In fact, many examiners tell the pupils to *assume* that the points are on a straight line, hoping to encourage motivation and immediate effort. (2) Again, even after the pupil decides to plot the points, he may not be sure which axis should carry the C-measurements and which those of D. He has been used to x and y in his previous tables. There y was, by convention, the dependent variable or the function letter. The y-axis was always the vertical axis. It is well to inform him that it does not matter in this case whether C or $D=y$. After all, a graph is a line showing the relation between two

variables. Freedom can be allowed in the choice of axes. Incidentally, a bright pupil brought to class lately the solution of the question under discussion. Though he had the final formula correct, his graph differed from the one drawn on the blackboard by the teacher. This puzzled him. After being told to turn his paper around and look at it between him and the light, he saw the graph conform to that on the board. (3) The next difficulty will centre around the initial equation. Should the pupil write $C=mD+b$ or $D=mC+b$ after discovering that the points in the table are on a straight line? That will depend on his solution of the difficulty in (2). He has already decided which letter$\equiv y$ and which x. (4) In the setting up of the equations necessary for solving m, b, the student may wonder which pairs of values to use from the table. He has been accustomed to use *all* the information given to him in previous work else something was wrong. Here, however, he must know (a) that more than two pairs are given so as to make quite sure that a straight line graph is determined and (b) that, as the same uniform law is assumed to run through all of them, it does not matter which pairs are selected. (5) In the actual solving of the two equations, both time is saved and the chance of error much lessened by the use of the addition or subtraction method. Pupils, however, may use other methods. Each must be sure, however, to select that mode of solution in which he is most proficient.

The importance and inherent possibilities of the topic will now be discussed.

A skilled and enthusiastic teacher can use it as a vantage point for both a backward and a forward view. (a) There is abundant opportunity here for remedial work in the different types of solution of simultaneous equations. The equation calls for the finding of the values of m and b. (b) Ample scope is afforded for practice in substitution—one of algebra's commonest processes. For example, in the solving of the equations, $6=2m+b$ and $8=5m+b$, after it is known that $m=2/3$, only *correct* substitution in either will give $b=14/3$. Then in the checking of the final equation from the table, the process of substitution will be necessary. (c) Absolute accuracy in the use of the fundamental rules is essential here and the importance of such will be brought home forcibly to the pupil. Wrong equations may be the result of one *small* error. (d) The proper manipulation of signs has ample scope for illustration in this particular field. (e) As a preparation for future *advanced* work the topic may be used to introduce the function concept—a wide theme. The formula, $C=mD+b$, shows clearly that any change (decrease or increase) in D brings about a corresponding change in C, provided m and b are constants. At once the pupil is on the outskirts of the calculus. (f) For later graph work, such terms as constant, independent variable, and dependent variable may be given a definite content at this stage. Later the student lets $y=x^2-x-6$, for example, in drawing the graph of that quadratic function. There y as the function letter or dependent variable will take on a clearer meaning. (g) Finally, $y=mx+b$ as the general equation of a straight line brings us to analytical geometry. Later he will learn that m is the slope and b the y-intercept.

How can the topic under discussion be made interesting? (1) The pupil can be shown how to supplement the work in the text by making a table of his own. This will prove to be a novelty and a real achievement. He selects any two letters, as e.g. Q and R. He models the table on those previously studied. Taking any fairly small number such as 4 for the initial value of Q he adds a definite fixed number each time to get 4, 6, 8, 10. Similarly for R, Starting with any value + or −, he gets −7, −10, −13, −16. So his table is:

Q	4	6	8	10
R	−7	−10	−13	−16

He will find that the points (4, −7), (6, −10) etc., are on a straight line. Therefore he can find the formula connecting Q and R just as we did for C and D. Moreover, the pupils can correct their own answers by checking from the tables they have made. When a class is busy making its own projects, we as teachers have accomplished something. (2) After a little practice, shortcuts in solution may be given. The value of m can be found by inspection. Divide the function-letter increase for any step by the corresponding increase for the other letter. In the above example, if $Q = mR+b$ then $m = 2/-3$ or $-\frac{2}{3}$. If $R = mQ+b$, then $m = -3/2$. The value of b can then be obtained by substituting in the equation $4 = -7\ (-\frac{2}{3})+b$ or in $-7 = 4(-3/2)+b$, according to the equation with which the pupil begins. Short cuts like these motivate the work. (3) The class may be asked to turn back to that section in their notebooks where graphs of the first degree equations in two variables have been drawn previously. There they have two straight lines intersecting. Somewhere on the squared paper they must have placed related values of x and y. Centre their attention on one of those tables. Let us suppose one of them is as follows:

$$\text{When } x = \quad 2 \quad 6$$
$$y = -2 \quad 1$$

Now ask them to obtain the original equation of one of the graphs before them which has already been drawn. The steps are easy. Starting with $y = mx+b$ (the reason is obvious), they set up equations as follows: $-2 = 2m+b$ and $1 = 6m+b$. So $m = 3/4$ and $b = -7/2$. $\therefore$ the original equation is $3x-4y = 14$. Interest is added because the topic is definitely related to work already finished.

In conclusion, we may state that this section in Second Course Algebra is one which can be made both interesting and useful in more ways perhaps than those indicated in this article. It certainly should not be regarded as detached or unimportant. The actual process of deriving a definite formula from a table of related values through a graph is one replete with mathematical possibilities.

Annual Meeting of the Mathematical Association of America!

THE twenty-fourth annual meeting of the Mathematical Association of America will be held at Columbus, Ohio, on Friday and Saturday, December 29–30, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the American Mathematical Society, and the National Council of Teachers of Mathematics.

The Mathematical Association will hold sessions on Friday afternoon and Saturday morning, with addresses by President W. B. Carver and Doctor Thornton C. Fry on the new journal, *Mathematical Reviews*, and papers on mathematical subjects by Professors Henry Blumberg of Ohio State University, Saunders Mac Lane of Harvard University, G. T. Whyburn of the University of Virginia, and Doctor E. F. Beckenbach of The Rice Institute. The annual business meeting and election of officers will be held at the beginning of the session on Saturday forenoon.

Professor J. R. Kline of the University of Pennsylvania, Vice-President of the American Association and chairman of Section A, will deliver his retiring address at 10:30 o'clock Thursday morning before a joint session of Section A, the Society and the Mathematical Association. His subject is "The Jordan curve theorem." The Association will meet jointly also with Sections A and E of the American Association and the Society on Friday morning, when papers will be given dealing with applications of mathematics to problems in geophysics, seismology and other earth sciences.

Two Related Units in the Teaching of College Algebra

By Zens Lawrence Smith
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When workers in such fields as astronomy, physics, and the like have occasion to refer to the binomial theorem they frequently if not invariably make use of the convenient "binomial coefficient" notation either in the form ${}_nC_m$ or, somewhat less commonly, in the form $\binom{n}{m}$. Such usage suggests a means of avoiding the awkward notation ordinarily employed when proof of the theorem for positive integers is adduced either in the second course in high school or in the freshman year in college. Furthermore, superior elegance in proof is by no means the only argument in favor of using the more convenient notation. As will be seen in what follows, proof by means of the binomial coefficient notation introduces the student to valuable and interesting experiences in fields other than that of the theorem itself.

In the field of elementary algebra there is probably no other group of closely related topics which catches the undergraduate imagination with such ease and force as the triple: probabilities, permutations, and combinations. Just why such stimulating subjects have of late years been pushed into the background in the usual undergraduate course is a question which does not call for discussion here. Those who have shed a tear or two of regret at the passing of these topics may be interested in these suggestions for their revival. In any event the reader, be his reaction sympathetic or otherwise, is asked to bear in mind that within the limits of a brief paper nothing more can be given than the barest outline of what requires from eight to ten class periods for anything approaching a full development. That this development is not at all "logical" is to the writer a source of pride rather than of chagrin.

Briefly, the procedure is as follows: The concept of probability is developed informally and a set of problems assigned in which are a few which involve simple permutations. These lead naturally to the desire for and subsequent development of the permutation formula. Mathematical induction is brought in here and is emphasized as a powerful method of proof. Factorials are also introduced as a convenient notation. Following the mastery of permutations further problems in probability are assigned. This time simple combinations are involved in such a way as lead to the combination formulas and to a few simple theorems. In particular, the class works out and preserves the theorem ${}_nC_{r-1}+{}_nC_r={}_{n+1}C_r$. The binomial theorem is then taken up after the method followed in most good texts but, as will be, seen in the fuller discussion which follows, the proof and the subsequent application of the theorem are greatly simplified by the use of the coefficient form, ${}_nC_m$.

The initial approach to probability is made by considering several practical situations which serve to develop the concept. For example, the instructor asks, "If I choose a member of this class at random, what is the probability that he will have previously attended the Morgan Park High School?" or "What is the probability that he lives on 11th Street?" In considering such questions the student is led to see what data are required as the basis for intelligent answers and is finally encouraged to formulate his own definition of probability, somewhat after the traditional fashion, as the ratio of the number of "favorable" cases to the number of all possible cases. It is perhaps needless to say that the deeply philosophical aspects of the question, such as those

centering around the concept of "equally likely," are not brought into the discussion.

When the concept of probability as a ratio has been well established and some practice has been given in selecting data for the solution of simple problems the instructor proposes some such question as the following: "If I choose three members of the class at random, what is the probability that I will select them in the order of their ages, the youngest coming first?" The same problem is then proposed involving four members of the class. For the moment nothing is said about permutations. The students are allowed to work out by rule of thumb, and without even a name for the process, such results as they will later come to handle quickly and conveniently under the formula for ${}_nP_n$. They will, of course, resort to various more or less awkward devices and will be the readier to accept with enthusiasm the instructor's suggestion that an investigation be carried out to see if some really effective way cannot be found for calculating these troublesome permutations. The name, "permutation," and its proper notation may now be effectively introduced and the class works out the formulas:

(1) ${}_nP_n = n!$

(2) ${}_nP_k = n(n-1)\ (n-2)\ \cdots\ (n-k+1)$

$$= \frac{n!}{(n-k)!}.$$

The development of these formulas affords opportunity for rich mathematical experience. In particular, the following points may be noted:

a. Mathematical induction may be introduced in developing the formula for ${}_nP_n$. (See Wilczynski's *College Algebra*, pp. 353–4.) This paves the way for use of induction is proving the binomial theorem.

b. By discovering the form of the kth term, the student receives practice in generalizing and so prepares himself for consideration of the general term when the binomial theorem is taken up.

c. The student is introduced naturally to factorials.

d. He learns to change a formula from a rather awkward form to a more compact one, although his attention should be called to the fact that the one which is formally more compact may not be the more convenient one to use in practice. This recurs in the development of the binomial theorem.

e. The instructor has an opportunity to emphasize once more that such a change as is indicated by the last two members of formula (2) involves nothing but multiplying the middle member by 1 in the convenient form $(n-k)!/(n-k)!$

During this excursion into the field of permutations certain problems in probability have been held in suspense. The class returns to these, solves them by means of the new tool, and now encounters something like the following: "If I select three members of the class at random, what is the probability that all three will be former students of the Morgan Park High School?" Again the students work in their own way until increasing complexity in the problems brings demand for a formal consideration of combinations. The permutation formula previously developed is used as a basis for working out the formula ${}_nC_m$ and enough problems are assigned to assure mastery of its use. The reasonableness of regarding ${}_nC_0$ as equal to 1 is brought out and the instructor also explains the feasibility of defining 0! as equal to unity. Just before returning to a final consideration of probability the instructor assigns the following as an honor problem: Prove that

$${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r.$$

Not all members of the class will succeed in solving this, but enough of them will get it to provide an interesting class discussion. When the theorem has been fully explained in class, all are asked to write out and hand in a proof that

$${}_nC_{k-2} + {}_nC_{k-1} = {}_{n+1}C_{k-1}.$$

The papers are returned and preserved in

the notebooks, the instructor observing that the theorem will come handy later on.

The unit is completed by using the new tools to attack certain remaining problems and theorems in probability; then attention is turned to the binomial theorem. A preview hints at the value of the theorem in solving compound interest problems, and in developing theorems of the calculus. The instructor then recalls to the class the method of induction already met in studying permutations. If some members of the class are still hazy in their understanding of induction, their difficulties should be thoroughly cleared up at this point. When these difficulites are resolved the theorem itself is approached by a study of $(x+y)^n$ for integral values of n from, say, 1 to 10. The instructor may show the class Pascal's triangle or, better still, its columnar form. The several expansions of $(x+y)^n$ are now studied and the following points noted:

a. Behavior of the exponents. Their sum is always n. That of x begins with n and decreases to 0, while that of y does the reverse. (There is a good opportunity here to review the meaning and use of the zero exponent.)

b. Behavior of the coefficients. They seem to take successively the forms ${}_nC_0$, ${}_nC_1, \cdots$, ${}_nC_{k-1}, \cdots$, ${}_nC_1$, ${}_nC_0$. (Here again opportunity is afforded for practice in deriving the kth term.)

c. Relations between exponents and coefficients. r in ${}_nC_r$ is also the exponent of y. Therefore, on account of (a), the exponent of x is $n-r$. In the kth term, the exponent of y is $k-1$, the right hand subscript of C is the same, and the exponent of x is $n-k+1$.

With these points in mind, the student has little difficulty in following the simple proof.

Proof. We know that for $n=1, \cdots, 10$ we can write

$$(1)\quad (x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + \cdots + {}_nC_{k-1}x^{n-k+1}y^{k-1} + \cdots + {}_nC_1xy^{n-1} + {}_nC_0y^n$$

Multiplying both sides by $(x+y)$ and combining like terms, we have

$$(2)\quad (x+y)^{n+1} = {}_nC_0x^{n+1} + ({}_nC_0 + {}_nC_1)x^ny + \cdots + ({}_nC_{k-2} + {}_nC_{k-1})x^{n-k+2}y^{k-1} + \cdots + ({}_nC_0 + {}_nC_1)xy^n + {}_nC_0y^{n+1}$$

But ${}_nC_0 = 1 = {}_{n+1}C_0$. Furthermore, when studying combinations we proved that ${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r$, whence (2) becomes

$$(3)\quad (x+y)^{n+1} = {}_{n+1}C_0x^{n+1} + {}_{n+1}C_1x^ny + \cdots + {}_{n+1}C_{k-1}x^{n-k+2}y^{k-1} + \cdots + {}_{n+1}C_1xy^n + {}_{n+1}C_0y^{n+1}$$

which shows that if the theorem holds for a positive integer, n, it holds for $n+1$. Complete induction follows at once.

The ease of applying the formula for either an expansion or the securing of a single term offers a decided advantage over the ordinary form. Let us say the student is asked for the sixth term of $(x+y)^{30}$. He starts by writing y with its proper exponent which is always $k-1$; in this case, 5. This is also the right hand subscript of C, the left hand being n. The sum of the exponents is n; therefore the exponent of x must be 25 in this case. In much less time than is required to describe the process, the student has written

$${}_{30}C_5\,x^{25}y^5 = \frac{30!}{5!25!}x^{25}y^5 = \frac{\overset{6}{\cancel{30}}\cdot 29\cdot \overset{7}{\cancel{28}}\cdot \overset{9}{\cancel{27}}\cdot \overset{13}{\cancel{26}}}{1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}}$$
$$x^{25}y^5 = 142506x^{25}y^5$$

Once the student is thoroughly familiar with his combination formula, he may omit the middle term, but in the beginning he will find it easier to write down all the steps, thus keeping his mind on one problem at a time.

The teacher who presents the binomial theorem to his class finds himself upon the horns of a dilemma. If he simplifies the presentation by omitting consideration of the general term, as is done in many high school texts, then his presentation is not a proof at all; if he does introduce the nth term, using the customary notation for the coefficients, the proof spreads out in such apparent confusion that many students dispair of ever grasping it. If presentation without proof is given, there is no particular advantage in using the "binomial coefficient" notation.

Reading in the Teaching of Mathematics

By Joseph B. Orleans
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During the past decade the complaint has become more and more frequent that the pupils in our high schools are deficient in reading. Various investigations have been conducted, a number of studies have been made and the range of reading ability in the ninth and tenth years in the high school has been found to dip down into the lower grades of the elementary school. Whatever are our opinions concerning this phenomenon, the fact remains that, under the present circumstances, we must accept the pupils who come to us, deficiencies and all, and we must do with them what we can.

In the teaching of mathematics the problem is aggravated by the fact that the text involves expressions composed of symbols, algebraic and geometric, in addition to and in conjunction with the ordinary English reading matter. To follow the path of least resistance and avoid reading situations, except when they are inevitable, is to evade a most important problem in the teaching of mathematics. For, if the pupils are properly trained in the learning of their subjects, they must resort to the textbook to supplement the activities of the classroom. To say that the pupils cannot read and, therefore, cannot study from the textbook by themselves does not meet the issue. The teacher of mathematics must train the pupil to read his text by using it in the classroom in connection with the presentation of new work as well as in drill and in review. Such training should be begun in the early years of the pupils' schooling. The development of material containing mathematical selections for silent reading will help. However, the untrained pupils are with us today. We must do something with them.

The following are sample lessons in which the open textbook is used in class. The pupils read silently, at times orally, and the reading is interspersed with oral questions and answers by pupils and teacher.

A. A Lesson on the Solution of a Pair of Simultaneous Equations.

Solve the pair of equations:

$$5x-y=19 \qquad (1)$$
$$7x+y=\ 5 \qquad (2)$$

Solution 1. Adding the equations, $12x=24$ or $x=2$.

Teacher: What happens to the terms containing y? How did the author get 24? How did he get $x=2$?

2. Substitute 2 for x in equation (1). Then $10-y=19$ or $y=-9$.

Teacher: Why did the author substitute 2 for x and not any other number? How did he get $y=-9$?

Check. In $5x-y=19$: Does $10-(-9)$ equal 19? Yes.

Teacher: Explain the answer "Yes." In $7x+y=5$: Does $14+(-9)=5$? Yes.

Teacher: How did he get 14? Explain the answer "Yes."

We get rid of y by addition in this example.

Teacher: Show where and how this addition is performed.

Solve the pair of equations:

$$5x+2y=11 \qquad (1)$$
$$x+3y=10. \qquad (2)$$

Solution 1. In order to get rid of y, you must make its coefficients have equal absolute values.

Pupil: What does this mean?

Teacher: The absolute value of a number is its numerical value irrespective of whether it is positive or negative. What is the coefficient of $+2y$?

Pupil: It is $+2$.

Teacher: Its absolute value is 2. If it were -2, its absolute value would still be 2.

2. Multiply equation (1) by 3.

$$15x+6y=33 \qquad (3)$$

3. Multiply equation (2) by 2. (4)
$$2x+6y=20$$

4. Subtract (4) from (3).
$$13x=13 \text{ or } x=1$$

Teacher: How was 33 obtained in equation (3)? Show exactly how $13x=13$ was obtained.

5. Substitute in (2). $1+3y=10$; $3y=9$; $y=3$.

Teacher: What did the author substitute in (2)?

Check. In (1): Does $5+6$ equal 11? Yes.
In (2): Does $1+9$ equal 10? Yes.

Teacher: How did the author get $5+6$ and $1+9$?

We got rid of y by subtraction in this example.

Teacher: Show where and how the subtraction was performed.

Elimination by addition or subtraction is the name of the method of solution taught above. Eliminate means "get rid of."

Rule: *If the coefficients of one of the unknowns have the same absolute value, eliminate that unknown by addition or subtraction.*

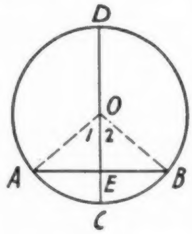
Teacher: How can you tell whether to add or to subtract?

If necessary, multiply one or both equations by such numbers as will make the coefficients of one unknown have the same absolute value. Then eliminate that unknown by addition or subtraction.

Teacher: In the equation $3x-2y=4$, what would you multiply by to make 10 the absolute value of the coefficient of the y? What would the new equation become?

B. Note: *The teacher and the class have together developed the proof of the following proposition without the textbook. It was done as an original exercise and the class "discovered" the conclusion under the guidance of the teacher. The pupils formulated the statement of the proposition. The teacher then announced, "You will study this proof for tomorrow. You will find it in your textbook on page . . . Now open your books to that page and let us read it together."*

If a diameter of a circle is perpendicular to a chord, it bisects the chord and its arcs.



Hypothesis. In $\odot O$, diameter $CD \perp AB$ at E.

Conclusion. $AE=EB$; $AC=CB$; $AD=DB$.

Plan. Prove AE and EB corresponding parts of $\cong\triangle$s. To prove $AC=CB$, prove $\angle 1=\angle 2$.

Teacher: When the author prepared the diagram for this proposition, what did he draw first?

Pupil: The circle.

Teacher: Then what did he draw?

Pupil: Chord AB.

Teacher: Then what did he draw?

Pupil: Diameter DC.

Teacher: How was the diameter drawn?

Pupil: Perpendicular to the chord AB.

Teacher: Then what did he draw?

Pupil: Radii OA and OB.

Teacher: Why are the radii OA and OB needed in the diagram?

Pupil: Because we need triangles AOE and BOE.

Teacher: What information can we get from the two triangles?

Pupil: By proving the triangles congruent, we can show that AE and EB are equal, and also that angle 1 equals angle 2.

Teacher: What will follow from the fact that AE equals EB?

Pupil: Arc AC will equal arc CB because equal chords have equal arcs.

Teacher: Comment on this statement.

Pupil: AE and EB are not chords of the circle.

Teacher: Then why are arcs AC and CB equal?

Pupil: Angles 1 and 2 are central angles that are equal and they intercept equal arcs.

Teacher: What follows from the fact that AE equals EB?

Pupil: If AE equals EB, then diameter CD bisects chord AB.

Proof	Statements	Reasons
1.	Draw OA and OB	....
2.	$\triangle OAE \cong \triangle OEB$	2. Give full proof.
3.	$\therefore AE = ?$	3. Why?
4.	$\angle 1 = \angle 2$	4. Why?
5.	$\therefore AC = ?$	5. Why?
6.	$CAD = CBD$	6. Why?
7.	$\therefore AD = ?$	7. Why?

Note: Pupils give the above reasons orally.

Thus the pupils are given practice in reading material involving mathematical symbols and expressions. The answers to the questions set by the teacher show whether or not the pupils have understood what they have read; and the discussion that intervenes the various parts of the reading matter helps in its interpretation.

The Christmas Party

By Phyllis A. Holroyd
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It was a snowy day with Christmas just three days away. The snow was falling fast in big soft flakes that were miniature hexagons.

A circle formed a center from which several streets led. On the curve of this circle stood a small square cottage. The windows were bright with holly circles and on the rectangular door hung another holly circle tied with a long piece of red ribbon. The cottage was the home of Miss Elvira Decimal, a little old lady, beloved by everyone in the little town of Mathematics.

This was the day for her annual Christmas party for the children of the Percentage, Fraction, Decimal, and Angle families. The big rectangular room was gayly decorated. A fire blazed in the square fireplace, over which hung a circular mirror. In the mirror was reflected a giant cone-shaped tree, standing in an angle of the room and brilliant with multicolored lights, and ornaments formed in squares, circles, and triangles.

The children came trooping merrily up the straight walk to the rectangular door where Elvira Decimal stood to welcome them. There were three of the Percentage children, Percentage, Rate, and Base; three Fractions, Decimal, Proper, and Improper; three Angles, Right, Acute and Obtuse; and the two Decimal children, Millionths and Tenths. Tenths was pulling a rectangular sled on which a square box was fastened. The sled was for Millionths, in which he was bundled snugly.

After the children had stood gazing in wonder at the beautiful tree, their mouths resembling various sized circles, their wraps were laid away. The little girls were wearing gay colored dresses of polka dots and striped material and the boys were dressed in equally bright colors.

Mrs. Unit, who lived but a few houses away on the opposite side of the circle, had come to help. She started the entertainment by teaching the children to play Chinese Checkers in which they put spheres in the holes on the star shaped board. After that they constructed little towns with rectangular solids, and they formed a circle to play games. They were then allowed to go out in the snow and make an immense circle and cut diameters to play fox and geese.

Later Mrs. Unit and Elvira Decimal led the children to a big rectangular table decorated with tall straight red cylinders set in the center of the little holly circles. The place cards for the Angle children were cut in angles suitable for their names and the others were made of cone-shape or squares and circles with a clever little drawing to represent each of the other children. There were served blocks of ice cream, delicious sandwiches and frosted cookies cut in squares, triangles, circles, rectangles, octagons, and stars, and they finished with big cylinders of chocolate milk in which ice cubes floated.

Gifts were presented to them. For the girls there were dolls, games, bracelets, rings, and beads, cut in squares, triangles and circles. The boys received wooden numerals, banks, cash registers, compasses, rulers, protractors, and baseball spheres. To all was given candy in little boxes of rectangles and squares, also oranges and apples that were big shiny spheres.

When the party was over and the Percentage, Fraction, Angle, and Decimal children were warmly bundled up and little Millionths was securely fastened in the square box, they started gayly down the street following in each others footprints in the snow and watching their

angular shadows formed by the big circular sun which was fading beyond the horizon.

As they reached the town's square which was already bright with multicolored lights, they saw a big round Santa Claus with his bright red suit and pyramid hat, standing by an immense cone-shaped tree. He was surrounded by numerous children and was giving each a box of candy. The boxes were bright little squares, rectangles, cones, and cylinders wrapped in cellophane.

Mrs. Unit and Elvira Decimal stood adjacent to each other at a small circular window and watched the children on their way. Elvira Decimal smiled through happy tears and said, "Such a little thing to do to make so much happiness. I am glad I can be a part of it."

A Merry Christmas and a Happy New Year to all of our Members!

Decimal-Form Fractions to Various Bases

By F. Emerson Andrews
Russell Sage Foundation, New York City

Fifty and more years ago, many common school arithmetics included exercises in numbers to various bases other than 10. Now the altered radix is usually reserved for hi~~r~~gher mathematics, and even the professional mathematician has often not proceeded in this field beyond the elementary processes.

The decimal in particular has been a source of confusion. We unfortunately use the word in two senses. Sometimes *decimal* means the system of expressing fractional quantities by whole numbers following a point; but sometimes the term is applied to the entire system of counting by tens. This has led many lay persons to assume that decimal-form fractions are the special prerogative of 10-system counting.

Mathematicians who have examined the subject at all, are aware that this is not the case. Decimal-form fractions are available in any number system which employs a zero and place-value. To the base 12, *.6* (meaning six-twelfths) is quite as adequate an expression for a half as the customary .5 (five-tenths). And *.4* (meaning four-twelfths) is an exact expression for one-third, a simple fraction which ordinary decimals are incapable of exactly expressing. Indeed, a table of the low fractions to various bases demonstrates how inefficient ordinary decimals are in comparison with several other systems of counting which might have been adopted except for the physiological accident of our ten fingers and thumbs.

Such a table is not difficult to construct. For its immediate interest, I present a sample, showing the low fractions from one-half through one-twelfth expressed in decimal form to the radixes II through XII.

Table I

Decimal-Form Fractions to Various Bases

Base	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
$\frac{1}{2}$	.1	$.11111\bar{1}$	.2	$.22222\bar{2}$	.3	$.33333\bar{3}$	.4	$.44444\bar{4}$	.5	$.55555\bar{5}$	.6
$\frac{1}{3}$	$.0101\bar{0}\bar{1}$	.1	$.11111\bar{1}$	$.1313\bar{1}\bar{3}$	.2	$.22222\bar{2}$	$.2525\bar{2}\bar{5}$	.3	$.33333\bar{3}$	$.3737\bar{3}\bar{7}$	.4
$\frac{1}{4}$	.01	$.0202\bar{0}\bar{2}$	.1	$.11111\bar{1}$	.13	$.1515\bar{1}\bar{5}$	.2	$.22222\bar{2}$	.25	$.2828\bar{2}\bar{8}$	.3
$\frac{1}{5}$	$.00\bar{1}10\bar{0}$	$.0\bar{1}210\bar{1}$	$.0303\bar{0}\bar{3}$	.1	$.11111\bar{1}$	$.\bar{1}25\bar{4}12$	$.\bar{1}46\bar{3}14$	$.1717\bar{1}\bar{7}$	.2	$.22222\bar{2}$	$.\bar{2}49\bar{7}24$
$\frac{1}{6}$	$.0010\bar{1}\bar{0}$	$.01111\bar{1}$	$.02222\bar{2}$	$.0404\bar{0}\bar{4}$	.1	$.11111\bar{1}$	$.1\bar{2}\bar{5}252$	$.14444\bar{4}$	$.16666\bar{3}$	$.1919\bar{1}\bar{9}$	.2
$\frac{1}{7}$	$.001\bar{0}0\bar{1}$	$.\bar{0}1021\bar{2}$	$.021\bar{0}2\bar{1}$	$.\bar{0}3241\bar{2}$	$.0505\bar{0}\bar{5}$	.1	$.11111\bar{1}$	$.125\bar{1}2\bar{5}$	$.\bar{1}4285\bar{7}$	$.163\bar{1}6\bar{3}$	$.\bar{1}86X35$
$\frac{1}{8}$	.001	$.0101\bar{0}\bar{1}$	.02	$.0303\bar{0}\bar{3}$	.043	$.0606\bar{0}\bar{6}$	.1	$.11111\bar{1}$	.125	$.1414\bar{1}\bar{4}$	.16
$\frac{1}{9}$	$.\bar{0}0011\bar{1}$	.01	$.013\bar{0}1\bar{3}$	$.\bar{0}2342\bar{1}$	.04	$.053\bar{0}5\bar{3}$	$.0707\bar{0}\bar{7}$	.1	$.11111\bar{1}$	$.\bar{1}2498\bar{6}$	.14
$\frac{1}{10}$	$.0\bar{0}011\bar{0}$	$.00\bar{2}20\bar{0}$	$.012\bar{1}2\bar{1}$	$.02222\bar{2}$	$.03333\bar{3}$	$.0\bar{4}620\bar{4}$	$.0\bar{6}314\bar{0}$	$.0808\bar{0}\bar{8}$	.1	$.11111\bar{1}$	$.1\bar{2}4972$
$\frac{1}{11}$	$.\bar{0}00101$ $110\bar{1}$	$.0\bar{0}211\bar{0}$	$.0\bar{1}131\bar{0}$	$.0\bar{2}114\bar{0}$	$.\bar{0}31345$ $242\bar{1}$	$.\bar{0}43116$ $235\bar{5}$	$.\bar{0}56427$ $213\bar{5}$	$.07324\bar{0}$	$.0909\bar{0}\bar{9}$	.1	$.11111\bar{1}$
$\frac{1}{12}$	$.0001\bar{0}\bar{1}$	$.002\bar{0}2\bar{0}$	$.01111\bar{1}$	$.0202\bar{0}\bar{2}$	.03	$.0404\bar{0}\bar{4}$	$.0525\bar{2}\bar{5}$	$.06666\bar{6}$	$.08333\bar{3}$	$.0X0X\bar{0}\bar{X}$	.1

Note 1. In this table single endlessly repeating decimals are indicated by a bar. Longer repetends are the numerals enclosed between two bars. Unclosed decimals are expressed to six points except in several cases under $\frac{1}{11}$, where the complete repetend requires ten places.

Note 2. Where it has been necessary in this table to express the quantity ten in a single place, the symbol *X* has been used in connection with Arabic numerals.

This table offers visual evidence of a few general facts about decimal-form fractions which are capable of logical proof, but are perhaps more readily seen in this form. It suggests certain relations between decimal and radix which are not so quickly observed in working with a single decimal system. For instance:

1. In any decimal-form system, a fraction can always be expressed as a whole

"decimal" if all the prime factors of its denominator are included among the prime factors of the number base.

Examples: One-half "comes out even" in all number systems with an even base, and is a repeating decimal in all others. One-third can be evenly expressed as a decimal only to bases 3, 6, 9, 12. . . . Conversely, because VI and XII contain the two lowest factors, 2 and 3, they express exactly more fractions in this table than any other number bases; indeed, they are not surpassed in this respect by any number base lower than XXX, which is far too large for convenient manipulation.

2. The number of figures in a decimal which "comes out even" is determined by the highest power to which a factor of the number base must be raised to include all the factors of the denominator of the fraction.

Examples: One-ninth $(3 \cdot 3)$ contains no factor not available in the base XII $(2 \cdot 2 \cdot 3)$, but requires the square of one of these factors. To the base XII it is therefore a two-place "decimal," *.14*. One-eighth is a three-place "decimal" to base II, runs to two places to base IV, and is expressed in one place to base VIII.

3. A fraction whose denominator includes a prime factor not found in the factors of the number base, cannot be expressed evenly to that base in decimal form. It will always resolve itself into an endlessly-repeating decimal.

A glance at the table demonstrates this rule. It often happens that this repetend runs just one place shorter than the denominator-numeral. That is, one-seventh has a six-place repetend to bases III, V, X, and XII. When this is the case, the repetend is a perfect circulatory number. Even when this is not the case, the repetend will be found to be a sub-multiple of one-less-than the denominator factor which is prime to the number base. This sub-multiple will have limited circulatory characteristics.

A circulatory (or revolving) number is one like 142857 (see one-seventh to the base X) which can be multiplied by any number from one through six and will reproduce exactly the same digits and in the same general order, but beginning at a different digit. Possibly the easiest way to discover such numbers is to express as a decimal fractions whose denominators are prime to the number base.

Such revolving or circulatory numbers are now being investigated in terms of primitive roots. They have been important in the history of number theory. It is possible that in them, particularly if they are explored in more than one number system for purposes of comparison, some of the ultimate secrets of Number may be discovered.

Curves

By ANNE W. YOUNG
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All curves derive from water: trouble it lightly,
A myriad ripples widen, circling like curtsies
Swept golden, black, pale blue, or sombre purple—
The deep is patterned like a field of Heartsease.

Clear paths of beauty, very plain determined
By an equation, brief prose describes their tether,
Though they be arcs fantastic tossed by fountains,
Emerald or brides-veil, floating hither and thither.

A wave breaks: once on a time Miranda's vision
Its line led ghostly through the reeds, or, flashing sudden
Above the rock, lifted on a chartless journey, free. . . .
A Source, betray not finally its plan deep hidden!

Our Mathematical Universe*

By H. G. Ayre
Western Illinois State Teachers College, Macomb, Illinois

We are living in a wonderful civilization, the greatest of all ages. The great learning and culture of the Greeks did not possess the dynamic characteristics of our twentieth century. The Roman civilization in all its pomp and splendor had no concept of our scientific age with its rapid transportation, communication, factories, great cities, and modern skyscrapers. The purple robed nobility of the middle ages, attended by scores of servants, knew nothing of our modern conveniences. The ancient kings ruled over vast domains and had great power and wealth, but there were no electric lights, no toasters, no sweepers, and no electric washing machines. Telephone, telegraph, and radio were unknown. Our age is spoken of as the scientific age, but probably it would be more nearly correct to call it the *golden age of mathematics*, for mathematics is the foundation of all science.

Let us fancy a picture—one so weird and ghastly that its existence as a reality would destroy the very civilization which we enjoy. Let us imagine that when we awaken tomorrow morning that there has disappeared from the earth all books on mathematics, all records pertaining to mathematics, every mathematical symbol, and every computing and recording machine. In the wake of such a catastrophe our loud speaker would be silent, the telephone would have lost its number, we could not describe the location of our homes in the city and the thousands of books in our libraries would have lost their identity. All forms of measurement—fine precision instruments, yard sticks, speedometers, electric meters, clocks, and calendars—would vanish. All building would cease; all business shut down; the wheels of industry come to a standstill. Banks, exchanges, and stock markets would be no more, aviation would be paralyzed, and ships at sea would be hopelessly lost, because all of these things owe their existence to mathematics.[1]

We could continue this ridiculous picture, but it probably has already revealed the universality of the role of mathematics in the development of a civilization.

In 66 A.D. a great light "hung over Jerusalem in the figure of a sword" and foretold the destruction of the city. One thousand years later a strange star appeared just before the conquest of England. In 1682 a bright comet was seen. Halley computed its orbit and behold, its path was an ellipse; therefore, had it been here before?—Yes! Records revealed that in 1607 Kepler had mapped exactly the same orbit. What about earlier records? Yes, history gave earlier appearances. It appears probable that the strange omen that gave warning of the destruction of Jerusalem and the conquest of England was Halley's comet. The great mathematician Clairaut found that it would continue to appear once every seventy-five years. Many of you remember that astronomers knew the exact time and place of its appearance in 1910. The comet is now on a long journey, but the exact hour when some of you will see it in 1985 is known.

A system of mathematics is the body of undefined elements, definitions, assumptions, and derived conclusions. The fundamental bonds of mathematics are *number* and *form*. Number implies integers, fractions, rational, and irrational numbers, and real and imaginary numbers. The relation of number to civilization proves

* Radio address delivered over station WTAD, Quincy, Illinois, 2:30 P.M., June 1, 1939.

[1] D. E. Smith, *The Poetry of Mathematics and Other Essays* (New York: Scripta Mathematica Library, 1934), pp. 16–17.

that man is naturally mathematical and he can't help it. Mathematics is the one universal language; it is a part of civilization. The early Egyptians developed enough mathematics to establish a calendar. It seems almost certain that the development of mathematics antedates the origin of the known languages.

Furthermore, there is some evidence that certain of the lower forms of life possess a kind of number sense. However, this peculiar faculty of number sense of the lower stages of development should not be confused with the higher mental process of counting. Man alone has mastered the higher processes so essential to exact thought, communication, and co-operative living.

When we consider *form*, the second of the fundamental bonds of mathematics, it is seen that in all plant and animal life there is a predominance of symmetry. An instinct of form is shown by some insects. It has been proved that the honey-bee encloses a maximum of space with a minimum of wax.[2] The numerous geometric forms in nature—the snowflake, crystal forms in minerals, the ammonite with the mathematically constructed spiral, the catenary in the spider web, the spiral nebulae, shape and motion of heavenly bodies—convinces one that nature is not haphazard or chaotic but governed by law and order. Whether on the earth or in the stars we find the same chemical elements, the same geometric forms, and the same fundamental laws.

All forms of growth, governed by constant environmental conditions, are represented by the same growth curve. The arrangement of leaves on the stem of a plant is represented by mathematical law. The divergence between leaves is shown by well-known number series. When the branch of a plant divides into smaller branches, it loses an energy which is imparted to the new branch. This relationship in energy is expressed by a formula involving trigonometry. Fechner astonished psychologists by showing that sensation and excitation in vision is described by mathematical law involving logarithms.[3]

Scientists tell us that the motion of the planets is like the action within an atom where there is the nucleus surrounded by electrons moving about as satellites, only the atoms are infinitely small. But the laws are the same. It is demonstrated everywhere that number has certain properties, geometric figures have definite relationships, the physical universe is governed by mathematical laws and our mathematical systems based on number and form are significantly connected with our physical universe.

Man no longer interprets the universe in anthropomorphic or mechanistic concepts. It now appears that nature is interpreted through concepts of pure mathematics.[4]

The pure mathematician is interested in creating a mathematical system that has a minimum dependence on experience. His work is sufficient unto itself—it is self-satisfying. Like the artist and musician he creates to satisfy an inner compulsion. He has little thought of elaborating a theory of utilitarian value. However, it is almost invariably true that these theoretical creations prove to be the means of explaining physical phenomena.

The Greeks studied regular polygons. This lead to much of the geometry which enables the astronomer to explain the motions of the planets. They developed the theory of conic sections. Archimedes used principles of conics when he focused the sun's rays on the ships in the harbor at Syracuse and set them on fire. The headlights of every automobile use the theory of conics developed by the early Greeks,

[2] George Wolff, "Mathematics as Related to Other Great Fields of Knowledge," *Eleventh Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1936), p. 225.

[3] *Ibid.*, pp. 221–222.

[4] Sir James Jeans, *The Mysterious Universe* (New York: The Macmillan Co., 1931), p. 143.

and an auditorium with good acoustic properties is built on a parabolic curve, so that the sound rays originating at a focal point on the stage are reflected from the walls in parallel rays.

Early in the sixteenth century Copernicus proposed the theory of a universe with the sun as a center. A hundred years later Kepler found that if circular orbits were assumed there was a discrepancy in the mathematics. However, he was familiar with the Greek's theory of conic sections and made the startling discovery that the paths of the planets are ellipses with the sun at one focus. Sir Isaac Newton and others formulated the law of universal gravitation and invented the calculus, a most powerful instrument for explaining the physical universe. These were the tools of Halley and Clairaut in determining the facts about Halley's comet.

For several centuries mathematicians tried to prove, or disprove, Euclid's parallel axiom. It was a little over a hundred years ago that this problem was settled by the creation of non-Euclidean geometry, which proved to be one of the most momentus events in the creation of abstract thought. Later this system of thought was further developed by creating a calculus based on the same assumptions. It was thought that this new mathematics was absolutely innocent of any practical applications, because its axioms seemed to contradict man's experiences in the physical universe. However, from 1906 to 1915 when the theory of relativity was being developed it was found that this new system of mathematics made clear new concepts of the physical world. Similarly, the recent theory of infinite classes seems to contradict the axioms of Euclid, but it has been found to agree with experience and furnish a medium of precise concepts of time and space.

Everyone listening to this broadcast knows that the use of copper wire is essential for the construction of his radio set, but many do not realize that the use of differential equations is equally as important. The whole wireless and radio industry of today had its inception in a few pages of mathematical analysis created by James Clerk Maxwell, who died about 60 years ago. Maxwell's electro-magnetic equations gave a very significant connection between light, sound, and heat, and led to the discovery of radio and wireless.

The physical universe is not governed by *chance* or explained by *superstition*. It is governed by *law* and *order* and its explanation is *inextricably* bound by *mathematical formulas*. Leverrier recognized this fact when in 1845 he undertook to account for the strange behavior of the planet Uranus. The orbit of Uranus had been calculated, but within a certain region she would leave her course. What could be the cause? No one knew. Leverrier connected this behavior with the laws of gravitation. Could there be another planet causing this disturbance? Observations were taken, computations made, results checked, more computations and more checking of results until finally Leverrier was sure he had found the exact spot in the sky where the planet was located. He wrote to his friend in the observatory at Berlin to set his telescope at a certain place in the sky and be the first to see a planet never before seen by mankind. The astronomer tried it out that night, and to his great delight and amazement his eyes were the first to greet the stranger, but he was very cautious and waited with anxiety and enthusiasm until the next night to make a check and found there was no mistake and the discovery of Neptune was announced to the world.

After this discovery it was found that Uranus had further disturbances that were unaccounted for. The fascinating story continues, and nine years ago at Flagstaff, Arizona, was discovered the planet Pluto. But yet there remains the question to challenge the best intellects in mathematics and astronomy—is there another planet beyond Pluto? Perhaps so. The depth of the future holds the answer.

Centuries ago the shepherd on the hillside of Judea looked into the skies and thought the stars were mere points of light all the same distance of a few miles away. He was so impressed by the spectacle that he exclaimed "The heavens declare the glory of God and the firmament showeth his handiwork." In civilization's debt to mathematics man sees countless bodies of vast magnitude, some as near as four light-years, then, star cities far into boundless space separated by millions of light-years, all governed by order, obeying every traffic signal of the vaulted skies, while traveling at unbelievable speeds. Yet, the mind that encompasses these things is only finite and cannot understand the infinite plan of the Great Architect. One can only gaze with wonder that deepens into awe and admiration as he joins the Psalmist in his words, "When I consider the heavens, the work of thy fingers, the moon and the stars, which thou hast ordained, what is man, that thou art mindful of him? and the son of man that thou visiteth him?"

Evolution of the Number Concept

To GLANCE at a single phase of the matter, what can be found in the whole history of thought more humanistically edifying than the story of the evolution of the number-concept, from the rudest beginnings in the primitive mind, long before men had learned even the first steps in the process of counting, to the great number-creations of the modern world? The concepts of Integers and Fractions, of Cardinals and Ordinals, of Positives and Negatives, of Rationals and Irrationals, of Reals and Imaginaries, of Algebraic and Transcendentals, of Finite and Infinites, these great concepts viewed with the occasions of their rise, with their struggles for existence, their ultimate triumphs over stubborn opposition, their persistent hardy growth through the centuries, their countless diversifications and subtle refinements, the infinite network of their interrelations and their manifold, always increasing, practical and theoretical uses and applications, afford a series of scenes that, for any one who has once contemplated them, constitute a truly, unforgettable, and inspiring panorama of the march of mind.—Cassius J. Keyser in "The Humanistic Bearings of Mathematics," *Mathematics in Modern Life*, Sixth Yearbook of the National Council of Teachers of Mathematics.

More Substantial Mathematics

UNTIL something more substantial than has just been exhibited, both practical and spiritual, is shown the non-mathematical public as a justification for its continued support of mathematics and mathematicians, both the subject and its cultivators will have only themselves to thank if our immediate successors exterminate both.—E. T. Bell, in a review of "The Poetry of Mathematics and Other Essays," by David Eugene Smith, *The American Mathematical Monthly*, XLII (1935), 559.

The Place of Mathematics in Secondary Schools*

By the Rev. William C. Gianera
University of Santa Clara, Santa Clara, California

Before entering into the discussion of the subject of this paper I wish to assure the members, and in particular the officers, of this organization that I consider it a distinct privilege to have been invited to express my ideas on this very important subject. Furthermore I wish to thank the officers for having acceded to my request in regard to the form that this discussion should take. From the moment that I was invited to address you I felt that a panel discussion of the subject would be far more profitable to everyone concerned than the mere enumeration of a long series of trite reasons and ideas as to why mathematics should be given a prominent place in the curriculum of our secondary schools, for I definitely believe mathematics should be given such a prominent place. I am in hopes that more benefit will be derived by all here present from the discussion which is to follow this paper than from the paper itself.

Naturally there will be some who will differ with me in regard to many of the ideas which I will set forth. I will not in any way feel hurt if such is the case. Perhaps even, I will touch on some things that will cause many of you to put me back in to the "horse and buggy" age of education as candidly I am not in accord with many of the developments that have come to present day education. My object in this paper and discussion is first to put before you what I feel should be the proper. place and importance of mathematics in the training of the future citizens of our country, second why I feel that way and third, since I believe that there has come about a break-down in the teaching of mathematics, what factors have to my way of thinking brought about this break-down. I am fully aware that there are many who will object to my statement about the break-down in the teaching of mathematics and all I can say to them is that I personally have definitely come to such a conclusion and I sincerely regret that the break-down has come.

At this point I would like to state that the ideas expressed here are drawn partly from theory, partly from observation and partly from discussion. First I say they are drawn partly from theory. By this I mean that after analyzing the concept of education and definitely establishing what true education should do for the individual being educated there should be established such a curriculum at the proper level that will best secure the object for which education is had. In other words some of my ideas are derived from an analysis of the objective value of mathematics as a preparatory training for future pursuits, whether those pursuits are to be academic or otherwise. This value of mathematics then is to be viewed not only for those who will go on to studies above the level of the secondary school but likewise for those who will terminate their academic work at the secondary school level and go into non-academic fields. Secondly I say from observation. Namely from my experience covering a period of eleven years in the admission of students to the University of Santa Clara. Unfortunately I have no figures to bear me out in this conclusion, yet I have nevertheless quite definitely observed that the quantity and quality of mathematics taken in high school is a good indication of the success the student will have in his college work. Moreover I have observed that even among students who do not like mathematics and who will avoid in college courses which will involve mathematics, provided they had mathe-

* **A paper read before the National Council of Teachers of Mathematics and the Department of Secondary Education of the N.E.A. at San Francisco, July 5, 1939.**

matics throughout their high school days, they will generally do better than those who were satisfied to fulfill the minimum mathematical college entrance requirements. Now I fully realize that this observation is not conclusive and I would like to add that this could be made into a very interesting and profitable study by one who is working for the Doctorate in the educational field. Thirdly I arrive at my conclusion partly from discussion. The discussion I speak of here has been in the nature of private conversations and arguments with various teachers both of college and high school levels in endeavoring to find out the reason why so many come to college unprepared for higher studies and without even the most elementary idea of analysis and concentration. Naturally in this latter respect my contacts have been very much of a local rather than of a national scope. The value of the sources above mentioned I rate in the order in which I have enumerated them.

A few words on what I include under the heading of secondary schools. I realize that even though the term "secondary school" is well defined and quite generally understood, particularly by those who have chosen teaching as a profession, the purpose of the secondary school is understood variously by the great number of educators. No one is ignorant of the various theories that have grown up regarding the object of education and the number of different curricula that have been drawn up to bring about the purpose or purposes of the secondary school education. The question now arises should there be one or many purposes in secondary education in view of the fact there are so many different elements to be taken into consideration. We all know that there are large high schools numbering their students in the thousands and there are the small high schools, there are city and rural high schools. Further we are aware that in all high schools there are the talented students, the mediocre students and the dull students. Moreover the students have a variety of likes and dislikes. Considering these elements and many others which have not and cannot be set forth here, educators have differed on the question of what should be taught. Better perhaps it is to say that educators have come to the conclusion that a variety of curricula should be offered to accommodate the greatly divergent conditions that have arisen in our secondary schools. They conclude that there cannot be a single purpose and consequently there cannot be any single method used. However, we must recognize that the period of education in the secondary school is a period of mental training or at least the period of mental training should begin in the secondary school no matter what type of school we have to deal with. Consideration should be given to the student more than to the school. Whether the student will go on to college work or not, be he talented or dull, this idea of mental training must always be of paramount importance. Lose sight of mental training in education and you lose sight of the principal factor in education. In brief, if mental training is set aside, education loses its very name. This idea does not do away with the distinction of academic and vocational schools, nor does it reject the idea that during the progress of education students should also acquire information. Neither is it true that everyone should be trained to go on to college. I am in sympathy with the idea of two general types of schools, academic and vocational, this latter used in a very comprehensive sense. Nevertheless even in these schools mental training should be aimed at as of prime importance. You may ask what is meant by this mental training. To me mental training is the developing in the mind of the student the power of analysis, concentration and adaptation of ideas, it is the being able to think. Hence any education which does not make the student think or at least fails to start the student thinking does not deserve the name of education.

With the foregoing in mind I feel free

to state that mathematics should hold a very prominent place in any curriculum of secondary education. In fact so important is mathematics in the mental training of the student that it should never be omitted. No other subject teaches the students to carry over principles and to combine and adapt those principles to a new set of facts and build up a unified structure. No other subject taught in our secondary schools today has the inherent qualities of making students think, analyze and concentrate, as does mathematics. No matter then what type of school we have, whether academic or vocational, irrespective of size, in view also of the varying degrees of mentality of the students, I feel that mathematics should be included in the curriculum of every student. Of course I do appreciate the fact that the study of English should come first and that for various reasons, but immediately following English should come mathematics. Today the brunt of mental development from secondary education has to be borne by the mathematics courses.

You will note that I said today the brunt of mental development has to be borne by mathematics. Of course as a true Jesuit still clinging to old ideals I sincerely believe that a study of the classical languages should by every standard be given a place even above mathematics. However, since the classics are so little taught except in the private secondary schools, I am purposely passing over their value in what I feel is the proper education of high school students. I regret very much that I have to admit this particular fact in regard to our secondary school education.

After English and mathematics are taken care of, then the informative courses can be added according to the circumstances; then, too, vocational work of the different fields may be given. For how many years should courses in mathematics be required? Since the period of the individual's training is to go on all during the secondary school stage, I would say that mathematics should be taught during the entire four years. Some will be prone to say that this is too much and impossible, not all students are capable of doing so much. The answer is clear, if the student is properly trained from the beginning and the teacher has the knack of drilling, and while drilling can make the courses interesting and point out the application of the principles, I believe that this program is not too hard to handle. True it is that not everyone is expected to do equally well. Division into groups according to ability will be necessary, nevertheless it will still remain true that the student taking mathematics will progress further in his training than if he goes on without it.

In this particular connection I wish to quote from Plato. In Book VII of his "Republic" he says: "And have you further observed, that those who have a natural talent for calculation are generally quick at every other kind of knowledge; and even the dull, if they have had an arithmetical training, although they may derive no other advantage from it, always become much quicker than they would otherwise have been." And later he adds, "And for all these reasons, arithmetic is a kind of knowledge in which the best natures should be trained, and must not be given up." In his work on "Laws," Book V, he makes this pertinent remark, "The legislator is to consider all these things and to bid the citizens, as far as possible, not to lose sight of the numerical order; for no single instrument of youthful education has such mighty power, both as regards domestic economy and politics, and in the arts, as the study of mathematics. Above all, arithmetic stirs up him who is by nature sleepy and dull, and makes him quick to learn, retentive, shrewd, and aided by art divine he makes progress quite beyond his natural powers."

To close this part of my paper I would like to call your attention to President Hutchins' work *The Higher Learning in*

America and in particular his chapter On General Education. Now I know that in the above mentioned book President Hutchins does not refer directly to the secondary school but what he says can be aptly applied to what we are discussing in this paper. Let me quote one paragraph: "Logic is a critical branch of the study of reasoning. It remains only to add a study which exemplifies reasoning in its clearest and most precise form. That study is, of course, mathematics, and of the mathematical studies chiefly those that use the type of exposition that Euclid employed. In such studies the pure operation of reason is made manifest. The subject matter depends on the universal and necessary processes of human thought. It is not affected by differences in taste, disposition, or prejudice. It refutes the common answer of students who, conformable to the temper of the times, wish to accept the principles and deny the conclusions. Correctness in thinking may be more directly and impressively taught through mathematics than in any other way. It is depressing that in high schools and junior colleges mathematics is not often taught in such a way as to achieve these ends. Arithmetic and geometry are there usually presented to the student as having great practical value, as of course they have. But I have had students in the freshman year in college who had never heard that they had any other value, and who were quite unwilling to consider mathematical questions until their practical possibilities had been explained. To this pass has our notion of utility brought us."

Now to come to the third point of this paper which I referred to above, namely, "what has brought about the break-down in the teaching of mathematics in our secondary schools." Personally I am inclined to set down two main causes which, to me, although they are general, are nevertheless fundamental. I am of the opinion that our system of education is wrong and that the method used in the teaching of mathematics is very much at fault.

To me the present day system of education in our secondary schools is an outgrowth of the elective system begun years ago and intended for advanced students, not for students in the lower grades. The elective system was to say the very least dangerous because of the abuses that were very apt to creep into education. The question was debated and discussed at great length at the time as undermining the very foundations of education. I believe the arguments against the elective system have been more than verified. The elective system gave rise to a great number of changes in our educational system, the principles involved gradually found their way into the secondary schools which in many instances became experimental laboratories for all the new ideas that were to be inaugurated by so many of our modern day educators. Little by little the true meaning of education was changed until today we wonder how the idea of mental training and mental discipline can be considered a part of the system. With the change in the idea of education it is not surprising that curricula should be changed. As I view the situation education today has no one thing that can be used as a definite norm to judge whether or not it is succeeding in doing what it is supposed to do. Although it will be denied by most modern educators that students are permitted to choose their own courses, I believe this to be true. I realize that perhaps our present day system considered in theory does not permit the students this liberty but the theory does not always work out in practice. Another point where our system today is wrong. I fear that more attention is paid to the so-called educational subjects in teacher training than to the subjects the teacher is expected to teach. To explain by an example; were I, as a principal of a secondary school, interested in engaging a teacher of algebra, geometry, etc., I think it should be more important that

the prospective teacher be questioned as to his knowledge of mathematics rather than if he understands the principles of educational psychology, whether he had a course in tests and measurements, how many units he has in the history of education, etc. Now mind I do not say that I am against these courses nor do I say that they are of no value to the teacher, but I do say that too much stress is put on them. I will go even further and say that although they can be useful and should be studied I do not believe that they are necessary.

My second cause for the break-down was that the method of teaching mathematics is at fault. First I feel that the teacher does not make known the educational or training value of mathematics. Certainly I believe that mathematics have a great practical value for the student in secondary schools, but more important than their practical value is their training value. Not only engineers and science men will agree with me on this point but also men successful in almost any field will bear me out. I refer here particularly to the lawyers. Recently a committee of the State Bar Association in studying the question as to what should constitute a good pre-legal education, wished as far as high school preparation was concerned, to set down among other things four years of mathematics, but because it was felt that this was too drastic they compromised by recommending at least three years of mathematics. Another point in the method of teaching mathematics, too much attention is given to the simpler problems, the so-called word problems from which I feel the greatest profit is to be derived are either not given at all or passed over very lightly. Again I feel that students do not like mathematics courses because they are not made interesting, the courses are made too dry due to the fact that instead of pointing out the applications of the various mathematical principles they are given a number of problems to do, mostly of the same type, which takes too much time and the study becomes altogether too monotonous. I believe that if classes in mathematics were made alive and interesting our high school students would get a different mental reaction towards the subject and would make progress far beyond what they are doing today.

This paper has already gone beyond the limits I had at first set down and since I believe the discussion is to be the profitable part of this meeting I think it time to close. I know that there are many other points that could be brought out here but I am in hopes that such points will come out in the discussion, if they do not perhaps it is just as well that they were developed here. I feel that anything of importance that has not been touched upon will be brought to light by those who will presently enter into this discussion. On the other hand what will not be touched upon will, in all probability, be relatively unimportant.

Save These Dates!

February 22 and 23, 1940—St. Louis, Missouri

Meeting with the American Association of School Administrators and its allied departments and organizations.

Theme: Mathematics for the "Other-than-College-Preparatory" Student.

Headquarters: Hotel Chase. (Get in your reservations at once.)

Program: Practically completed, a really *National* meeting, to be announced soon.

June-July 1940—Milwaukee, Wisconsin

Theme, headquarters and program not yet determined. Suggestions welcomed.

Suggestions and questions should be directed to E. A. Katra, Executive Secretary N.C.T.M., 525 West 125th Street, New York City.

General Mathematics for Art Pupils*

By Alma Ekholm
Girls Commercial High School, Brooklyn, New York

In Girls Commercial High School three courses are offered: academic, art and commercial. The academic girls take the regular mathematics: algebra, geometry, etc. The commercial girls take one term of commercial arithmetic in their second term. The art girls take one year of general mathematics in their first year. The introduction of this course for the art pupils was influenced by the emphasis upon the social objectives of the "school of today" and by the principles of educational psychology. The course aims to overcome the prejudice against mathematics which pupils advancing to high school invariably feel and to give them a broad view of the field of the subject. Mathematics is correlated with art and other subjects whenever possible. Instruction is related to the present.

The course begins with a unit of graphs, correlated with civics and science. Interesting graphs of budgets are made. The graph showing how the government suggests a dollar should be spent on food appeals to the girls. We have just graphed "The World's Fair Dollar." Pupils enjoy illustrating their work.

This is followed by a unit of intuitive plane and solid geometry (of size, of form, and of position). By inspiring sidelights we keep the girls interested. When teaching perpendiculars, I refer to the late Professor Pupin's first lesson in geometry which he described when he was guest speaker at one of the dinner meetings of The First Assistants of Mathematics in New York. To emphasize the difference between perimeter and area, the story about the present size of the City of Carthage is told. Thus the girls also realize the need for the system of measures and that the custom of using animal skins was not reliable. We introduce Roman and Gothic architecture. The girls usually bring in extra homework: e.g., the architecture of their places of worship. Locus is introduced through *Treasure Island* which the pupils read in their first term.

The subject about volumes is begun with the words of Professor Hedrick of the University of California (when he was the speaker at one of the dinner meetings of Section 19, New York Society for the Experimental Study of Education), "A tall can on the grocer's shelf is a monument to woman's ignorance of mathematics." A graph is made showing the increase in volume of a cylinder when the height is increased and the radius is kept constant and on the same set of axes, the increase in the volume when the radius is increased and the height is kept constant.

The art girls thoroughly enjoy scale drawing. They delight in sketching their dream houses and drawing the rooms to scale. Formulas are introduced through the use made in clinics for babies' feedings. Equations and also directed number are taught. Trigonometry is included. A year ago last summer one of the girls wrote to me from camp that the Girl Scouts were using this subject.

In the second term after a presentation of the intuitive proof of the Pythagorean Theorem, which we call a jigsaw puzzle,

* A paper read before a panel of a joint Conference of the Mathematics Chairmen's Association and the Association of Teachers of Mathematics of New York City at the Hotel Astor, March 11, 1939.

dynamic symmetry is taken up. This is a system of inter-related rectangles, possessing common properties used by the ancient Egyptians and then by the Greeks who applied Euclidean geometry. Only recently the late Professor Jay Hambidge rediscovered it. The basic figures taken up are: root rectangles, reciprocal root rectangles, whirling square and similar constructions. Keen interest is shown when the pupils learn that the dragon fly and the growing iris fit into the root three rectangle. When the spiral is presented, the pupils appreciate the law and order of the universe. They are surprised to learn that the following, though so different, conform to the logarithmic spiral, namely: the path of a rolling coin, the arrangement of the sun-flower seeds, water running out of a tub, the snail shell, the closed hand, etc. I believe that the easiest way to construct this spiral is by means of the whirling equilateral triangle. In the second term our girls read the *Odyssey*. The edition which they use has illustrations showing spirals and frets. The girls are on the alert for examples of dynamic symmetry. A few days ago one of the girls, Miss Cutrone, came rushing into the room with the announcement that Abraham and Straus, a department store in Brooklyn, N. Y., had new blouses on display with a poster reading, "Blouses featuring dynamic symmetry."

As this is the last term the art pupils have mathematics a unit of commercial mathematics is included. I call this unit, "Protective Mathematics." It has been said that Americans seldom count their change and rarely verify columns of figures when purchases are made. Professor Bell of the California Institute of Technology has said, "At the high school level, every prospective citizen should learn enough about the arithmetic of simple investment to make him or her immune to the shady schemes that put money into the pockets of the few at the expense of the many."

To motivate fire insurance, maximum insurance for minimum premium is discussed. Girls surprise their parents with this fact and with the knowledge that two clauses appear in their policies.

We discuss life insurance sold by fraternal organizations, companies and New York State savings banks (since January, 1939). A short history of the 1907 Armstrong investigation is given showing how Chief Justice Hughes and former Justice Brandeis made their reputation when they proved that the reserve money belongs to the insured. Examples are worked showing that two mathematical laws have prevented failure of life insurance companies: law of probability and law of compound interest. Though some companies sell as many as eighty-four different kinds of policies, only the four basic types are explained. Examples are worked to discover the value of each type. It is shown that the endowment policy is most expensive and is not for a young person, nor advisable for a person with dependents. Premiums on endowment policies and similar deposits in savings banks are contrasted. When studying compound interest the girls bring in their bankbooks.

"How many girls have parents who send money abroad?" is the question which motivates the lesson on commercial banks. Many raise their hands. All parents seem to send money orders. Last September at the time of the European crisis when the foreign exchange was falling rapidly, was an excellent time to teach foreign drafts.

Stocks are bought "on paper" and girls are quite excited about their choice and anxiously figure their daily loss or gain.

Before the topic on the cost of a one family house versus apartment is taught, the girls vote that the house is less expensive. Later they are surprised to learn that the cost of a house is so high.

The next topic: finding the rate of interest charged when buying on the installment plan and by loan companies is so important that I wish we could teach it to all our pupils. Frequently questionable

schemes are exposed. I know of a teacher who borrowed money from a loan company. She believed that she was paying the same rate that banks pay on deposits, 2% per annum. She did not realize that she was paying 2% per month. Moreover, she believed that she had a special rate, as advertised for teachers.

A unit of demonstrative geometry is included to give pupils the knowledge of a proof, perhaps the only logic they will have. Moreover, artists recognize the fact that they need geometry. Leonardo da Vinci always regretted that he had not studied mathematics when he was young. In Valentin's book, he quotes from da Vinci's manuscripts (page 244) "Let no man who is not a mathematician read the elements of my work" and (page 243), "No human inquiry can be called science unless it pursues its path through mathematical exposition and demonstration." You may be interested to know that da Vinci's errors were due to mistakes in dealing with fractions (page 21).

I quote from a modern article which appeared in *The Technology Review* of February, 1938, by Rutherford Boyd, who designed and directed the film, "Parabola." "For the designer there remains concealed in the postulates and axioms of Euclid, the basic principles of abstract form and its great qualities of rhythm, symmetry and preparation!"

The Adventures of Little An Alysis*

An Alysis was a little girl who didn't like to work;
She didn't like to go to school, and her studies she would shirk;
She didn't like her English, and she hated history;
But of all the studies she liked least, she despised geometry.
She couldn't tell a circle from a rhombus or a square;
And she thought that area was a word pertaining to the air.
She thought hypotenuse was a beast owned by Pythagoras.
In fact, she didn't know a thing. She was the dumbest lass!

Now An Alysis had a cousin whose name was Poly Gon,
And Poly was the smartest girl you'd find in Oregon.
She knew a tangent was a line that touched a circle once;
And, since she knew her axioms, she had never been the dunce.
Now Poly knew her cousin An was very dumb indeed.
And thought she'd help her out, since she was so much in need.
She told her first of angles, the obtuse and acute;
She taught her how to make an arc, and how to find square root.
She taught her just a lot of things, and in the end, you mind,
Our little An Alysis was the brightest you could find.

—Written by Carol Culver in a plane geometry class at Iowa State Teachers College

EDITORIALS

The National Council's New Executive-Secretary

MRS. FLORENCE BROOKS MILLER, who has for many years now been Chairman of the State Representatives of the National Council of Mathematics, has seen fit to give up this work on account of the large responsibilities it entails. Under her careful and wise guidance this work has become a job for someone who can give a considerable amount of time to it. Since at the last meeting of the Council at Cleveland in February of this year, it was decided to employ a part-time executive-secretary who might later become a permanent officer, this work has been turned over to him.

The Council is fortunate in having secured Mr. A. E. Katra of the University of Illinois High School to fill this new position, and he is already at work in the office of THE MATHEMATICS TEACHER, 525 West 120th Street, New York City. He is starting out with great interest and enthusiasm, and THE MATHEMATICS TEACHER wishes him the greatest success.

Mr. Katra, in addition to taking over Mrs. Miller's work, will assist the President of the Council in arranging programs, securing speakers, and in looking after them at Council programs. It is also certain that such an industrious and versatile person as Mr. Katra will find many other useful things to do.

But we should not want to end this editorial without expressing our deep sense of gratitude for the fine work Mrs. Miller has done in past years in trying to build up the membership and friends of the Council.

W. D. R.

Possible Improvement in Mathematics

IT SEEMS reasonably clear that teachers of mathematics in many places throughout the country fail to make as much progress in modernizing their courses as they should because of a feeling that the gap between their practice and that of the more liberal programs is too great for them to bridge. While this situation may be true, it is also possibly fair to say that such teachers forget that they do not need to go the full distance the first time they resort to change. Moreover, changes in time-honored teaching material should not be made blindly, but surely some gain can be made by picking out some areas where necessary improvements are obvious. By working on these things some good reorganization of material can be accomplished. For example, many teachers know that certain material in algebra and geometry is not only valueless at certain stages in the pupil's career, but is obsolete as well, and yet little is done about it. If mathematics is to continue to hold its high place among the other great fields of knowledge, we must begin, if only in a small way, to bring about some necessary changes not only in subject matter but in methods of teaching.

W. D. R.

Reform in the Schools

IN THE APRIL 1939 issue of THE MATHEMATICS TEACHER, we commented on an article by Professor James L. Mursell, of Teachers College, Columbia University, on "The Defeat of the Schools." We believe that the readers of the TEACHER will be interested in a follow-up article by Professor Mursell in the December 1939 number of *The Atlantic* on "The Reform of the the Schools." Whether mathematics teachers agree with what is said here is not the important issue. The main point is for us to see that mathematics is improved as much as possible in any reform that is made.

W. D. R.

Sixth December Meeting of the National Council of Teachers of Mathematics

To Be Held Jointly with the Mathematical Association of America
Columbus, Ohio, December 28 and 29, 1939

General Theme: Relational Thinking in Secondary School Mathematics

**Thursday, December 28, 6:30 P.M.,
University Faculty Club**

I. JOINT BANQUET WITH THE ASSOCIATION AND THE SOCIETY

For reservations write Dr. S. E. Rasor, Professor of Mathematics, Ohio State University, Columbus, Ohio

**Friday, December 29, 9:30 A.M.,
Derby Hall, Room 100**

II. TRAINING TEACHERS FOR RELATIONAL THINKING

Presiding: F. L. Wren, Peabody College for Teachers, Nashville, Tennessee

Training Teachers of Arithmetic for Relational Thinking. J. T. Johnson, Chicago Normal College

Training Teachers of Junior High School Mathematics for Relational Thinking. C. C. Richtmeyer, Central State Teachers College, Mt. Pleasant, Michigan

Training Teachers of Senior High School Mathematics for Relational Thinking. H. G. Ayre, Western Illinois State Teachers College, Macomb, Illinois

Training Teachers of Junior College Mathematics for Relational Thinking. J. S. Gorges, Wright Junior College, Chicago, Illinois

**Friday, December 29, 9:30 A.M.,
Derby Hall, Room 102**

III. RELATIONAL THINKING IN SECONDARY MATHEMATICS AS VIEWED BY THE COLLEGE TEACHER

Presiding: R. L. Morton, Ohio University, Athens, Ohio

Speakers: F. L. Wren, Peabody College for Teachers, Nashville; Albert A. Bennett, Brown University, Providence, Rhode Island

Other Panel Members: Charles Weidemann, University High School, Columbus, Ohio; James R. Overman, Bowling Green State University, Bowling Green, Ohio; H. C. Christofferson, Miami University, Oxford, Ohio

**Friday, December 29, 2:00 P.M.,
Derby Hall, Room 100**

IV. JOINT MEETING WITH THE MATHEMATICAL ASSOCIATION

Presiding: H. C. Christofferson, Miami University, Oxford, Ohio

Speaker: Dr. Karl Bigelow, Director of Commission on Teacher Education, American Council on Education

Panel Discussion of Implications of Dr. Bigelow's Address

From the point of view of the Liberal Arts Colleges: B. W. Jones, Cornell University

From the point of view of the State Universities: H. H. Downing, University of Kentucky, Lexington, Kentucky

From the point of view of the Teachers Colleges: W. O. Shriner, Indiana State Teachers College, Terre Haute, Indiana

From the point of view of the Association and Council: Edwin G. Olds, Carnegie Institute of Technology, Pittsburgh, Pennsylvania

**Friday, December 29, 2:00 P.M.,
Derby Hall, Room 102**

V. TEACHING CHILDREN TO DO RELATIONAL THINKING

Presiding: Harold Fawcett, University High School, Columbus, Ohio

Speakers: In Junior High School Arithmetic: C. L. Thiele, Supervisor of Exact Sciences, Detroit, Michigan

In Algebra: Dale Wantling, University High School, Columbus, Ohio

In Geometry: Marie Sangernebo Wilcox, George Washington High School, Indianapolis, Indiana

Other Panel Members: Dorothy Wheeler, Bulkeley High School, Hartford Connecticut; F. N. Harsh, Principal High School, Kent University, Kent, Ohio; A. Brown Miller, Shaker Heights,

Ohio; Alma Wuest, Walnut Hills High School, Cincinnati, Ohio

Teacher Training Program Committee:
- F. L. Wren, Peabody College for Teachers, Nashville, Tennessee
- L. H. Whitcraft, State Teachers College, Muncie, Indiana
- R. P. Agnew, Cornell University, Ithaca, New York

Secondary School Mathematics Program Committee:
- Mary A. Potter, Racine, Wisconsin
- Ralph Beatley, Harvard University, Cambridge, Massachusetts
- Harold Fawcett, Ohio State University, Columbus, Ohio

Local Steering Committee:
- Harold Fawcett, Ohio State Unifersity, Columbus, Ohio
- Charles Weidemann, University High School, Columbus, Ohio
- John Ramseyer, University High School, Columbus, Ohio

Exhibits Committee: Charles Weidemann, Chairman
- T. F. Maloney, North High School, Columbus, Ohio
- John F. Schacht, Bexley High School, Columbus, Ohio
- Karl A. Zeller, Central High School Columbus, Ohio
- F. O. Williamson, East High School, Columbus, Ohio
- H. L. Zulauf, West High School, Columbus, Ohio

Registration Committee:
- John Ramseyer, University High School, Columbus, Ohio

Headquarters: Neil Hall, lodging at University dormitories and fraternity houses. For reservations write Mr. John Ramseyer, University High School, Ohio State University, Columbus, Ohio

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IN OTHER PERIODICALS

By Nathan Lazar
The Bronx High School of Science, New York City

1. Christofferson, H. C., "Teaching Functional Relationships in Elementary Algebra," *School Science and Mathematics*, 39: 611–617. October, 1939.

After reviewing some of the literature which emphasized the importance of functional relationship in the teaching of elementary algebra, the author proceeds to a detailed presentation of nine different types of problems in which the notion of dependence predominates. "The idea of determining the relationship which exists between quantities with which we deal, is one of a fundamental mathematical nature. It is one which is interesting to children and it is one which should produce large dividends in terms of mathematical appreciation, mathematical reasoning, and mathematical insight."

2. Dawson, T. R., "Match-Stick Geometry." *The Mathematical Gazette*, 23: 161–168. May, 1939.

An interesting study of the mathematical possibilities of the familiar "match-puzzles."

The geometer is supposed to be provided with an unlimited supply of "match-sticks," that is finite straight lines, all of one length L. He constructs figures with these lines according to four postulates which are stated. It is claimed that "the methods of construction postulated are capable of determining all points obtainable with rules and compasses, but no others."

3. "Geometry, the Second Report on the Teaching of." *The Mathematical Gazette*, 23: 169–184. May, 1939.

A report of a discussion that took place at the Annual Meeting of the (British) Mathematical Association, January 3, 1939. The following participated: Siddons, Hooke, Bearwood, Kenworthy-Browne, Inman, Sheppard, Kearney, Snell, Roebuck, Broadbent, and Fletcher.

4. Jones, B. W., "On the Training of Teachers for Secondary Schools." *The American Mathematical Monthly*, 46: 428–434. August-September, 1939.

The writer believes that "the course of training for a prospective teacher of mathematics in secondary schools should differ from that for one going on to research in the following fundamental ways: first, it should be broader; second, it should lay chief emphasis on thorough understanding; third, it should give by the close of the first graduate year material for mathematical investigation as well as training in it. The standards, though different, must be just as strict."

In answer to those who disparage all "lower" instruction, the writer says: "We must realize that there are many ways to serve mathematics: advance its frontiers, recruit its army, spread its propaganda. They require different aptitudes, but who shall say that one is more important than the other!"

5. Menger, Karl, "On Necessary and on Sufficient Conditions in Elementary Mathematics." *School Science and Mathematics*, 39: 631–642. October, 1939.

In the history of mathematics there are examples of confusion between necessary and sufficient conditions which seemed to hinder the development of some branches of higher mathematics. Although the research of our times maintains a degree of logical rigor which precludes any such elementary blunders, the textbooks, and presumably the teachers of mathematics, too, still perpetuate such errors.

A very clear explanation and analysis is given of several problems in algebra, geometry, and the calculus in which a sufficient condition is mistaken for the necessary, and a necessary condition confused with the sufficient one.

6. Orleans, Joseph B., "Testing the Ability of Pupils to Read in Mathematics," *High Points*, Vol. 21, No. 7, September, 1939, pp. 18–24.

It is generally admitted that very few of our pupils refer to the text for help, and, if they do, they do not succeed in getting accurate knowledge because they have not been trained to read the mathematical material even in the elementary textbooks. The writer believes "that the teaching of mathematics in the high schools can be improved immeasurably through the use of the textbook in the classroom, a procedure which would require the pupil to study through reading under the guidance of the teacher."

A detailed list is given of questions in alge-

bra, geometry, and trigonometry which were included in the uniform examination so as "to test the ability of the pupils to read and study new material by themselves from the printed page, and in order to evaluate the effectiveness of the emphasis placed in the classroom upon developing this ability."

In this connection, see below the summary of the article by Scarlet.

7. O'Toole, A. L., "An Approach to Trigonometry." *National Mathematics Magazine*, 13: 373–375. May, 1939.

Although many textbook writers and teachers begin with the general formulas for the functions, they often place too much emphasis on certain right triangles in setting up the definitions of the functions. It is recommended that it would be better to put the major emphasis on a point on the terminal side of the angle, its coordinates x and y and the distance r from the origin to the point.

8. Scarlet, Will, "Reading in Mathematics." *High Points*, Vol. 21, No. 4, April, 1939, pp. 26–34.

"Gone are the days when mathematics teachers concerned themselves solely with the formal aspects of their subject. . . . To a great extent, the materials of instruction in mathematics are still expressed in a dusty verbalism smacking of the classical tradition. . . . Many mathematics teachers themselves are disturbed by the opaqueness which marks so much textbook material. . . . If reading efficiency in mathematics is tc be promoted, textbooks must be prepared by persons who are not only competent in the field but who are aware of general and specific reading deficiences of secondary school pupils. . . . Just as english teachers are beginning to isolate the types of words and sentences which tend to confuse students, so should mathematics teachers undertake to describe the incidence of error for the various reading skills in mathematics."

The following specific suggestion is made: "On your next departmental examination, whether it be in algebra or geometry, include a question or two which requires the pupils to do intensive reading of mathematical material. These reading questions should not call upon them to demonstrate any computational skill whatsoever. . . . Require the pupils to answer pointed questions which will test their skill in comprehension."

9. Schaaf, William L., "Testing the Clarity of Mathematical Concepts." *School Science and Mathematics*, 39: 651–656. October, 1939.

As every teacher knows, students may use words or phrases whose meaning has apparently been mastered although actually the concept in question is only vaguely grasped or ill understood. In order to reveal and evaluate the degree to which pupil-mastery of the concepts has been achieved two projects are presented: (a) a list of 94 mathematical terms which are reasonably important and commonly encountered, most of which are verbal symbols for fairly abstract or generalized concepts, and (b) a suggested short-answer type test of 60 statements, including some of these terms, the aim of which is to reveal to the teacher the degree of mastery of those terms, or at least, which ones are inadequately understood.

"The arbitrariness of the terms chosen is granted and no claim is made as to the reliability or validity of the tests. If it proves a stimulus for further study, our object shall have been accomplished."

A useful bibliography of twelve references is included.

10. Tuckey, C. O., "A Diagram for the Study and Solution of Triangles." *The Mathematical Gazette*, 23: 150–154. May, 1939.

The diagram resembles, in principle, the nomograms used to solve equations of various types. It may be used to solve the various cases of triangles, to find the area of a triangle, to discover the range of variation of one element of a triangle when other elements are given, and to solve certain problems in mechanics.

"It is hoped that in the near future a complete large-scale diagram will be available for purchase by those interested."

11. Wolff, Georg, "The Second Report on the Teaching of Geometry." *The Mathematical Gazette*, 23: 185–197. May, 1939.

The main objection to the report lies in its failure to urge teachers to get a little further away from the Euclidean spirit.

NEWS NOTES

The Men's Mathematics Club of Chicago and the Metropolitan Area met for the first time this season on October 20. Dr. Harold C. Taylor was the speaker of the occasion and took for his topic "Applications of Mathematics in Personnel Research." Dr. Taylor is Chief of the Psychological Research Section, Hawthorne Works, of the Western Electric Company.

The Women's Mathematics Club of Chicago and Vicinity held its first meeting of the year October 7. Miss Marie Plapp gave the first of a series of five minute talks by members of the club. Her topic was "Mathematics in the New York World's Fair." Luncheon was followed by a tour of the Rosenwald Museum of Science and Industry. A member of the staff gave a brief explanation of the exhibits in the museum that are of particular interest to mathematics teachers.

The second meeting of the Women's Mathematics Club of Chicago and Vicinity was held in Chicago's "China Town" at Won Kow Restaurant November 4. A five minute talk was given by Miss Edith Levin of Englewood High School, speaking on "The Uses of Mathematical Devices in Teaching." Doctor Philip Fox, director of the Rosenwald Museum of Science and Industry, was the guest speaker, taking for his subject "Mathematics in Science and Industry."

DOROTHY B. LANDERS

The Tulsa County, Oklahoma, teachers met in October for annual election of officers, a short business session, and a program of important addresses. Aubrey Henshaw presided, and C. L. Gordon acted as secretary.

"Securing and Maintaining Interest in Teaching Mathematics" was the subject of a talk by Raymond R. Scott. Avo Davis discussed "Important Changes in the Tulsa Mathematics Curriculum." D. L. Barrick was also a speaker.

The Forty-sixth Annual Meeting of the American Mathematical Society will be held at Columbus, Ohio, Tuesday to Friday, December 26–29, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the Mathematical Association of America and the National Council of Teachers of Mathematics. The sessions of the Society will begin Tuesday afternoon and continue through Friday morning. The Mathematical Association will hold its sessions Friday afternoon and Saturday morning, and National Council will hold sessions Friday morning and afternoon. All sessions will be at Ohio State University.

The fifteenth Josiah Willard Gibbs Lecture will be delivered Wednesday afternoon by Professor Theodore von Kármán, Director of the Daniel Guggenheim Aeronautical Laboratory of the California Institute of Technology. The title of this lecture is *The engineer grappling with non-linear problems.*

The Board of Trustees will meet at noon on Wednesday and the Council of the Society will meet at 7:30 P.M.

The annual business meeting and election of officers will be held Thursday morning and at this time the award of the Frank Nelson Cole Prize will be announced. Following this session, Professor J. R. Kline of the University of Pennsylvania, Vice-President of the A.A.A.S. and Chairman of Section A, will deliver his retiring address before a joint session of the Society and Section A. Professor Kline's title is *The Jordan curve theorem.*

On Thursday afternoon, Professor D. H. Lehmer will give an invited address entitled *The Application of Bernoulli Polynomials to Some Problems in Diophantine Analysis.*

A joint dinner of the mathematical organizations will be held on Thursday at the Faculty Club.

A joint meeting of the Society, Section A and Section E is being planned for Friday morning to discuss applications of mathematics to geological and geophysical problems.

Applications are being received for Benjamin Peirce Instructorships in Mathematics at Harvard University for the academic year 1940–41. Candidates should ordinarily have the doctorate or its equivalent. Applications should be sent to, and further information may be received from, the Chairman of the Department of Mathematics.

Dr. S. Ulam has been appointed Lecturer in Mathematics at Harvard University for the academic year 1939–40.

The Extension Division of the University of Iowa conducted its Fourteenth Annual Con-

ference of Teachers of Mathematics, on October 13 and 14.

Some of the speakers and the titles of their addresses follow: Mary A. Potter, Supervisor of Mathematics, Racine, Wis.—"Arithmetically Speaking" and "In Defense of Donald the Dull"; Grace Dean Maynard, Theodore Roosevelt High School, Des Moines, Iowa—"Creative Teaching"; Allen T. Craig, University of Iowa—"Mathematical Systems"; W. S. Schlauch, New York University—"Mathematics as an Interpreter of the World and Life" and "Mathematics Useful in Life"; Alice Hatch, Fort Dodge High School, Fort Dodge—"Capitalizing on the Individual Differences and Interests in the Junior High School"; H. K. Newburn, University of Iowa—"Observations on Mathematics in English Secondary School."

The play "The Case of Matthew Mattix" was given by pupils of the University High School under the direction of Ruth Lane. The motion picture "The Play of the Imagination" was also shown.

Professor Ralph Beatley, of Harvard University, addressed the Fall Meeting of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England on November 4. His subject was "Teaching the Beginning of Analytic Geometry."

Other speakers were Professor Haroutune Dadourian of Trinity College whose subject was "What Is Truth?—The Authoritarian, the Mathematical, and the Scientific Concepts of Truth," and Professor Bancroft Brown of Dartmouth College who spoke on "Gambling for Pi."

Professor W. D. Reeve of Teachers College, Columbia University addressed the first meeting of Section 19 (Mathematics) of the New York Society for the Experimental Study of Education at its first meeting of the new year on "The Problem of the Gifted and the Dull Normal Pupil."

Index photographed at the beginning for the convenience of the microfilm user.